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DETERMINING SHEAR ELASTICITY MODULUS BY TORSIONAL PENDULUM

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Abstract: *When clamping the upper end of a thin bar or wire and fixing a circular plate to the bottom end of it, we get torsional oscillating system. The shear elasticity modulus can be calculated after determining the lowest root of the characteristic equation of the second order differential equation describing the torsion oscillation of the linear bar, and measuring experimentally the smallest natural frequency of the torsion oscillation.*

1. INTRODUCTION

As it is known, the material equation of homogenous isotropic materials contains three characteristic data: E the coefficient of elasticity, G shear elasticity modulus and ν the Poisson's ratio. If one know two of them, the third can be calculated by the formula

$$E = 2(1+\nu)G \; . \qquad 1.1$$

Poisson's ratio is determined from relative longitudinal and cross-directed elongations measured on a tensile-test piece tensioned. Metrological problems arises when measuring these elongations of wires under 1mm in diamater applied in stranded wire. Because of this reason determining Poisson's ratio is possible by equation 1.1, if E and G can be measured experimentally. The coefficient of elasticity E can be obtained from tensile test, the shear elasticity modulus from torsion oscillations. This is the object of the next chapter.

2. DETERMINING SHEAR ELASTICITY MODULUS

2.1 Theoretical background

We get torsional oscillating system if we clamp the upper end of a thin bar or wire, which the diameter is d and length is h of, and fix a circular plate to the bottom end of it (Figure 1.)

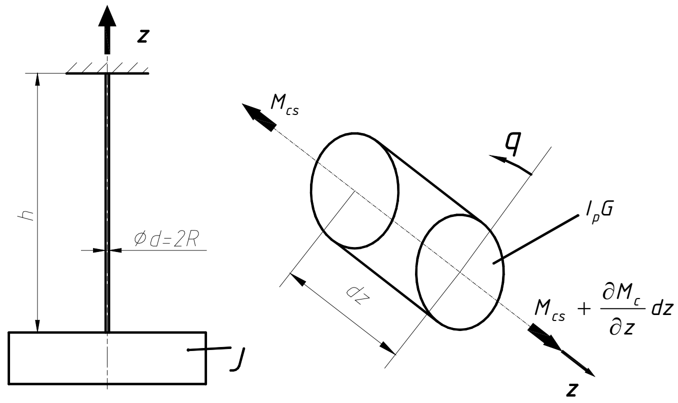

Fig. 1. *The scheme of a torsional oscillating system*

Eqwuation of motion of torsion oscillation of a linear bar can be derived from the principle of torque and angular momentum applied to dz thick a slide of it :

$$dJ_z \ddot{q} = \frac{\partial M_{cs}}{\partial z} dz, \qquad 2.1$$

where q generalized coordinate denotes the angular deflection around the centroidal axis of the cross-section, dJ_z is moment of inertia of the elemental piece of the bar concerning to the axis z:

$$dJ_z = \rho \int R^2 dA dz = \rho I_p \, dz, \qquad 2.2$$

I_p is polar moment of inertia of the circular bar

$$I_p = \frac{d^4 \pi}{32} \qquad 2.3$$

and ρ is the mass density of the material of the bar.

Supposing elastic deformation of the wire, and using the relation 2.4 between the relative angular deflection and the torsional load

$$\frac{\partial q}{\partial z} = \frac{M_{cs}}{I_p G} \qquad 2.4$$

we can write equation (2.1) in the form

$$\rho I_p \frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial z}\left(I_p G \frac{\partial q}{\partial z} \right) \qquad 2.5$$

where G denotes the shear elasticity modulus of the bar.

While the cross section is constant we get the equation of motion in the following form:

$$\frac{\partial^2 q}{\partial t^2} - a_1{}^2 \frac{\partial^2 q}{\partial z^2} = 0 \qquad 2.6$$

which is a second order partial differential equation. We introduce for the ratio of G and ρ :

$$a_1{}^2 = \frac{G}{\rho} \ . \qquad 2.7$$

If we suppose that oscillations of all cross sections of the bar have the same α angular frequency, the the solution of the differential equation can be obtained as a product of two function, one of them is function of coordinate z, another of them is of time t (Fourier-method):

$$q(z,t) = \varphi(z)\cos(\alpha t + \varepsilon)\,, \qquad 2.8$$

where φ(z) is interpreted as amplitude distribution along the bar.

The characteristic equation can be written taking into account boundary conditions (clamping and presence of circular plate)

$$-J\alpha a_1 \frac{1}{I_p G}\sin\left(\frac{\alpha}{a_1}h\right) + \cos\left(\frac{\alpha}{a_1}h\right) = 0 \qquad 2.9$$

where J denotes the moment of inertia of the plate concerning to the axis z.

Introducing the notation

$$\beta = \frac{\alpha}{a_1}h \qquad 2.10$$

we can write the coefficient $J\alpha a_1/I_pG$ int he formula 2.9 as

$$\frac{J\alpha a_1}{I_p G} = \frac{J a_1{}^2 \beta}{I_p G h} = \frac{J}{h I_p \rho}\beta = \lambda\beta \qquad 2.11$$

where

$$\lambda = \frac{J}{h I_p \rho} \ . \qquad 2.12$$

Inserting this into the characteristic equation we get

$$\lambda\beta = \mathrm{ctg}\beta \ . \qquad 2.13$$

This shows that roots of characteristic equation β_1, β_2, … can be obtained from intersections of graph of line $y=\lambda\beta$ and curve $y=\mathrm{ctg}\beta$ (Figure 2).

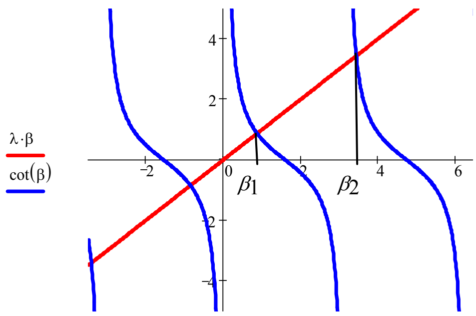

Fig. 1. *Solutions of characteristic equation*

It can be shown, that for large values of λ (when the moment of inertia of the bar concerning to the axis of rotation $J_r = hI_p\rho$ is smaller than the moment of inertia of the circular plate J), we can approximate the smallest normal frequency of the system by the following formula:

$$\alpha_{min} \cong \sqrt{\frac{I_p G}{h\left(J + \frac{1}{3} J_r\right)}} \quad . \qquad 2.14$$

2.2 Realization of the torsion oscillating system

To ensure the same boundary conditions, the ends of the wire was soldered into the clamping heads shown in Figure 3. Centralization of the wire was guaranteed by the hole 1 mm in diameter in the head, and another hole with same diameter in the circular plate.

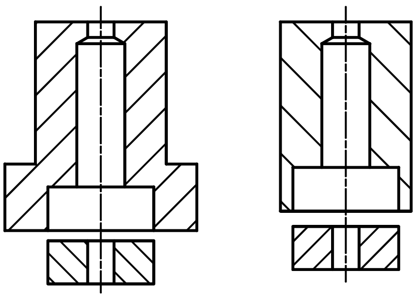

Fig. 2. *Clamping heads*

The mass, serving the moment of inertia fixed to the bottom end of the wire is consisted if two parts: cylinder with hole (1) and cylindrical plate (2). The cylindrical plate and the head are clamped by headless screw (3). Cylinder is clamped to them by the screw 3. (Figure 4.) . The upper end of the wire was clamped by a drillstock linked to a screw spindle M20. The length of the wire was measured by an altimeter.

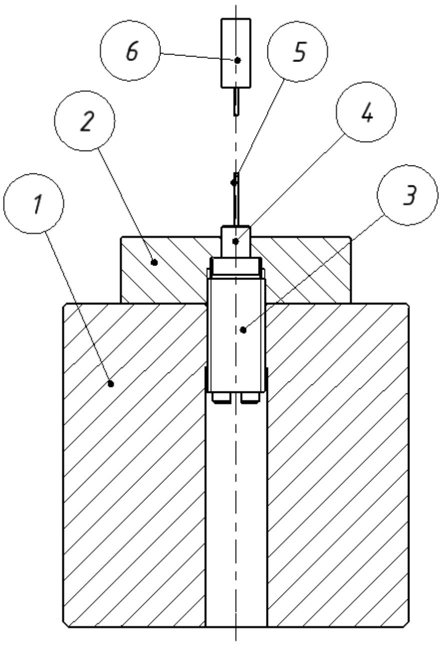

Fig. 3. *The shape of the weight*

Fig. 4. *The assembly of the torsional pendulum*

2.3 Parameters of the system

2.3.1 Parameters of the weight:

General data:

Mass density: 7,860E-006 kg/mm^3

Volume: 1,704E+005 mm^3

Mass: 1,339 kg

Moment of inertia: J = 584,895 kg mm^2

Centre of mass:

X: -9,507E-016 mm

Y: 2,092E-016 mm

Z: 31,302 mm

2.3.2 Parameters of wires investigated

Parameters of wires investigated are shown in the table below,

S.sz.	d	A	h	ρ	I_p	σ_z	J
	[mm]	[mm^2]	[mm]	[kg/mm^3]	[mm^4]	[Mpa]	[kgmm$^{2]}$
1	0,68	0,363168111	359,6	8,576E-06	0,020991117	36,1571	6,47351E-05
2	0,68	0,363168111	451,55	8,576E-06	0,020991117	36,1571	8,12879E-05
3	0,38	0,113411495	351,9	8,209E-06	0,002047077	115,783	5,91349E-06
4	0,38	0,113411495	431,1	8,209E-06	0,002047077	115,783	7,2444E-06

where the polar moment of inertia: $I_p = \frac{d^4\pi}{32}$, the diameter of the wire: $A = \frac{d^2\pi}{4}$, the axial normal stress int he wire: $\sigma_z = \frac{Q}{A}$, the moment of inertia: $J_r = \frac{1}{32}\rho\pi h d^4$.

In our experiments parameters of macaroni with circular cross-section available in commerce were measured. While the cross-section differed from the circle considerably parameters of 10 piece were averaged.:

d_1	d_2	A	h	ρ	I_p	σ_z	J_r
[mm]		[mm^2]	[mm]	[kg/mm^3]	[mm^4]	[Mpa]	[kgmm$^{2]}$
2,7665	0,9822	2,50049	341	1,448E-06	5,65936	5,251404553	2,79438E-03

where the polar cross-sectional moment of inertia: $I_p = \frac{(d_1^4 - d_2^4)\pi}{32}$, the mass moment of inertia: $J_r = \pi h\rho\frac{(r_1^4 - r_2^4)}{2} = \frac{1}{2}m(r_1^2 + r_2^2)$.

Mass moments of inertia of wires (and macaroni) are negligible compared to the moment of inertia of the weight fixed to the end of them in formula 2.14.

2.4 Investigation of torsional oscillation

In our series of experiment the same wire was measured with two different lengths. Time of 20 period was measured in two series. In first series we performed our measurements with different starting deflections ($\varphi_0 = 10°,20°,30°,40°$). This does not affects detectably the experimental results of time of period. Of course, the amplitude decreased in time, especially int he case of macaroni. Because of this only time of 10 periods was possible to measure of macaroni. Averaged experimental results are summarized in the following table:

Type of wire	Number of periods measured	Time measured		length	diameter	Time of one period	Angular frequency
	n	T_n	$T_{averaged}$	H	d	T	α
	[db]	[s]	[s]	[mm]	[mm]	[s]	[1/s]
steel wire I. series	20	39,05	39,0550	359,6	0,68	1,9528	3,21761
steel wire II. series	20	39,06					
steel wire III. series	20	43,81	43,8075	451,55			
steel wire IV. series	20	43,81				2,1904	2,86854
zither string I. series	20	123,14	123,1475	351,9	0,38	6,1574	1,02043
zither string II. series	20	123,16					
zither string III. series	20	139,83	139,8275	431,1			
zither string IV. series	20	139,83				6,9914	0,89871
macaroni I. series	10	8,10	8,0375	341	d_1=2,7665		
macaroni II. series	10	7,98			d_2=0,9822	0,8038	7,81734

2.5 Evaluation of results

The quantity λ expressing the gradient of the line is function of the mass of weight m and its mass moment of inertia (I_p), the material properties of the wire (ρ), the length (h) and diameter (d) of it.

Type of wire	h	λ	β	α	a_1	G	G_{approx}
	[mm]		rad	[rad/s]	[1/s]	[Mpa]	[Mpa]
Steel wire	359,6	9,035E+06	3,325670E-04	3,21761	3,479155E+06	1,038083E+05	1,037358E+05
Steel wire	451,55	7,195E+06	3,726680E-04	2,86854	3,475723E+06	1,036036E+05	1,035311E+05
zither string	351,9	9,891E+07	1,005500E-04	1,02043	3,571260E+06	1,046967E+05	1,046963E+05
zither string	431,1	8,074E+07	1,112920E-04	0,89871	3,481219E+06	9,948393E+04	9,948468E+04
mavaroni	341	2,093E+05	2,185780E-03	7,81734	1,219570E+06	2,153685E+03	2,15369E+03

where $\lambda = \frac{J}{hI_p\rho}$, β is the smallest solution of equation 2.13 , $a_1 = \frac{\alpha}{\beta}h$.

The shear elasticity modulus expressed from 2. 11 we get $G = \rho\left(\frac{\alpha h}{\beta}\right)^2$, and G_{approx} approximate value calculated by approximating formula $G \cong \frac{hJ}{I_p}\alpha^2$.

3. CONCLUSION

Determining the time of period of the torsional oscillation can be disturbed by two main factors. One of them is the coupling between the torsional oscillation and swing oscillation. The other of them is the normal stress coming from the reaction force of the weight.

Values of G_{approx} obtained by the approximating formula shows good aggreement with G calculated from our experimental measurements. Tis is especially true int he cases of zither stroing and the macaroni.

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