THE FRACTAL TECHNIQUES APPLIED IN NUMERICAL ITERATIVE METHODS FRAME

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Abstract: *Many methods in the numeric calculus resolving nonlinear algebraic or differential equation but ignore some behavior of these problems. Same aspects can conjure to the fractal iterative techniques. In this paper work is performed a critical analysis about iterative fractal techniques which can conduct to various implemented applications. Study of the nonlinear equation, treated into iterative techniques, makes the subject of this paper. It consists in a short revue of the most important principles of the fractal calculus and complexity applications in fundamental sciences and technologies. If trying to solve the equation* $z^4-1=0$ *in the complex plan, we can obtain the Newton's fractal. And this is not the single case when a numeric method for solving nonlinear algebraic equations has same strange behavior. If we modify the fractal logistic equation with the goal to perform an appropriate model for some physical applications, we can obtain some interesting and amazing results.*

Key words: *fractal, attractor, map, set, iterative, complexity, form.*

1. INTRODUCTION

Many methods in the numeric calculus resolving nonlinear algebraic or differential equation but ignore some behaviour of these problems. Since fractal analysis developing were identified a lot of strange features in the solution of same equations.

A very interesting phenomenon occurs in the solution of the following set of nonlinear differential equations called the Lorez system [5]:

$$\frac{dx}{dt} = -10(x-y) \quad \frac{dy}{dt} = -xz + rx - y \quad \frac{dz}{dt} = xy - \frac{8}{3}z \tag{1}$$

This system arises from problems related to fluid convection and to weather forecasting. When the r parameter lies in the $24.7<r<145$ interval, the solution does not converge to a fixed point in the $t\rightarrow\infty$ limit, nor is there a limit cycle, but the solution keeps moving around in a finite area. The limit set of the orbit at $t\rightarrow\infty$ is generally called the attractor. It has been confirmed numerically that the Lorenz attractor system has infinitely many folding.

Other strange attractors have been found in many systems with few degrees of freedom. The following system, called the Rössler system [5], is famous for showing that chaos can be produces with only one nonlinear term:

$$\frac{dx}{dt} = -(y+z) \quad \frac{dy}{dt} = x + 0.2y \quad \frac{dz}{dt} = 0.2 - 5.7z + xz \tag{2}$$

Attractors of ordinary differential equations with the degree of freedom less than 2 are limited to either a fixed point or a limit cycle, and have proved not to be strange. However, even in system with only two variables, chaos can be found if the system evolves discretely. A good example in this sense is the strange attractor of the Hennon map. The equations system in this case is:

$$x_{n+1} = 1 - ax_n^2 + by_n \quad y_{n+1} = x_n \tag{3}$$

Strange attractors in systems of ordinary differential equations also usually have fractal properties. By imagining a plane in the phase space and observing only the points where the orbits pass through the plane, the dynamical systems can be reduced to a discrete map called the Poincaré map. The Poincaré map of the Rössler system, like the Henon map, is self-similar and the Rössler attractor is also fractal.

2. STRANGE ATTRACTORS

Let us consider a simple nonlinear map called *logistic map* or *bifurcation map*:

$$x_{n+1} = rx_n(1-x_n), \quad 0 \le r \le 4 \tag{4}$$

This is an example of iterative method application on nonlinear function. In the first regard it is a classical and very knowledge path to resolving without problems same numeric analysis applications. However, we can observe that the asymptotic behaviour of x_n depends strongly on r parameter:

for $0\le r <1$, x_n decrease as n and x_n approach 0; for $1\le r \le 2$, x_n monotonically approaches $1-1/r$; for $2< r \le 3$, x_n approaches $1-1/r$ with oscillations; for $3< r \le 3.449$, x_n is gradually approaches period motion of period 2; for $3.449< r \le 4$, the system become uncontrolled.

The set of attractors of x_n is shown in figure 1.

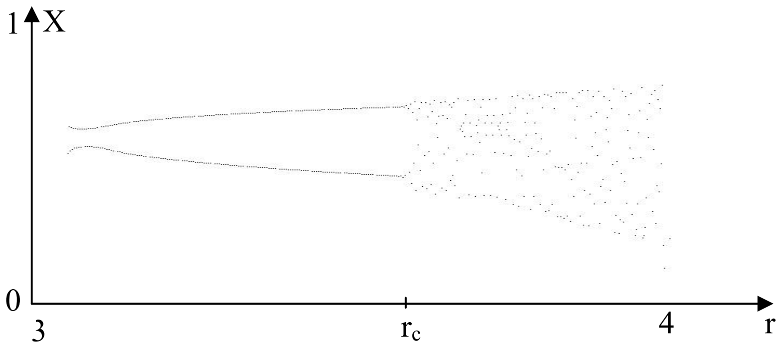

Fig. 1 *Bifurcation diagram of logistic map for $3 \le r \le 4$*

Historically, the logistic map was obtained from the logistic equation, which describes the growth of a population in a closed area:

$$\frac{du}{dt} = (\varepsilon - hu)u \tag{5}$$

If we put this equation into a difference equation forms:

$$\frac{u_{n+1} - u_n}{\Delta t} = (\varepsilon - hu_n)u_n \tag{6}$$

we obtain the logistic map if we change the variables as:

$$r = 1 + \varepsilon \Delta t \quad x_n = \frac{h\Delta t}{1 + \varepsilon \Delta t} u_n \tag{7}$$

The solution of (5) can be obtained analytically for any initial condition $u(0)>0$. It monotonically approaches a fixed point ε/h. By contrast, the difference equation for large interval Δt and the logistic map behave quite differently, producing chaos. This kind of discrepancy between the solution of a differential equation and that of its difference equation appears in any nonlinear system if the difference interval is sufficiently large [1]. Hence we have to be careful when we numerically solve a differential equation by using a difference equation.

If we modify the logistic equation in the form:

$$x_{n+1} = rx_n /(1 - x_n), \quad 0 \le r \le 4 \tag{8}$$

we can observe an interesting result about the map equation (figure 2). This result makes subject of some original studies focused on the numeric methods in complexity calculus.

If trying to solve the equation $z^4-1=0$ in the complex plan, we can obtain the Newton's fractal [4] which shown like in figure 3. And this is not the single case when a numeric method for solving nonlinear algebraic equations has same strange behaviour.

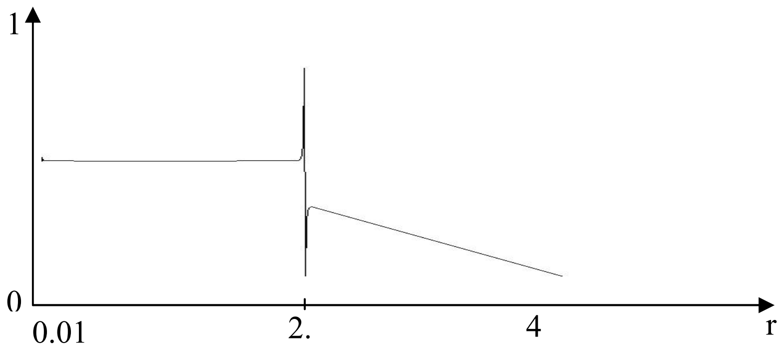

Fig. 2 *Bifurcation diagram of modified logistic map for 0.01≤r≤4*

3. FRACTALS BY MAPS

For a given map [2]:

$$x_{n+1} = f(x_n) \tag{9}$$

the set of initial points $\{x_o\}$ whose iterated points never diverge ($|x_n|<\infty$ for any n) is called *Julia set*. For many maps, the *Julia sets* are known to be fractals. A good example is the following complex logistic map:

$$f(z) = az(1-z)\text{, where } z=x+jy, j=\sqrt{-1}\ . \tag{10}$$

In the same way, equation:

$$g(z) = z^2 - b \tag{11}$$

conducts to other fractal. To set of complex parameters b such that successive iterates of $z=0$ under $g(z)$ do not tend to ∞ is named the *Mandelbrot set*. This set has a fractal border.

When we solve an algebraic equation numerically by Newton's method, we have to iterate a map similar to (11). If the equation has several solutions, an initial value for the iteration will be attracted to one of the solution. The boundary of the set of points that finally converge to one of the solution becomes a fractal. Two initial points that are arbitrarily close can approach distinct solutions, if they start close to this boundary.

Another simple method to construct fractals is provided by *contraction maps*. It is trivial that the invariant set of a single contraction map is a point. However, for two or more contraction maps the invariant set is the set X which satisfies:

$$x = f_1(x) \cup f_2(x) \cup ... \cup f_n(x) \tag{12}$$

this is a fractal. For example, in the case $n=2$, the following maps produce the Cantor set in the [0,1] interval.

$$f_1(x) = \frac{x}{3},\quad f_2(x) = \frac{2+x}{3} \tag{13}$$

In the complex plane we have the Koch curve if the mappings are:

$$f_1(z)=\alpha\bar{z},\quad f_2(z)=(1-\alpha)\bar{z}+\alpha,\quad \alpha=\frac{1}{2}+\frac{\sqrt{3}}{6}j \tag{14}$$

where $\bar{z}$ denotes the complex conjugate of z.

Thus all regular (non-random) fractals can be expressed in this formalism, which because its simplicity is expected to become more important in future.

4. RANDOM CLUSTERS

Consider a 2 or 3 dimensional lattice and distribute points randomly on it with p probability. If neighbouring sites are occupied by points, they are regarded as connected. By changing the probability p of the occupation of sites we can estimate the critical probability p_c and fractal dimension of clusters.

The fractal dimension of clusters is calculated in the following way. We define the mean radius of clusters of size s as:

$$R_s = < (\sum_{i=1}^{s} \frac{r_i^2}{s})^{1/2} > \tag{15}$$

where r_i denotes the distance between the centre of mass and the i-th point, and <•> indicates the average over all s-clusters. When R_s is proportional to a power of s, the clusters are statistical fractals with dimension D which satisfies the relation:

$$R_s \sim S^{1/D} \tag{16}$$

The result of simulations show that (16) holds at $p=p_c$ and the fractal dimensions are estimated as 1.9 (2 dimensional lattice) and 2.5 (3 dimensional lattice) [3]. This value in the 2 dimensional cases agrees with the experimental value.

5. CLUSTERS IN SPIN SYSTEMS

The best-know model of magnetic material is the *Ising model* [4]. In this model, spins which can take only the value +1 or -1 are arranged on a lattice. The total energy (or Hamiltonian) E of the system is given by the equation:

$$E = -J\sum\sum S_i S_j - H\sum_i S_i,\quad S_i = \pm 1 \tag{17}$$

Here, $\Sigma\Sigma$ denotes summation over nearest neighbour sites. J is the coupling constant and H is the external field. In thermal equilibrium, the probability of occurrence of the state with total energy E is given by:

$$W \sim e^{-E/k_B T} \quad (18)$$

where k_B is Boltzmann's constant and T denotes temperature.

In both 2 and 3 dimensional space, the Ising model is known to show a phase transition at a critical temperature, T_c. For $T<T_c$, symmetry is spontaneously broken and most spins take the same value, which indicates that the system is ferromagnetic. On the other hand when $T>T_c$, each spin takes the value +1 or -1 nearly independent of neighbouring spins and the average of spin vanishes, which shows that the system is demagnetised. At the critical point $T=T_c$, the characteristic size of clusters of the same spin diverges and distribution of the clusters becomes fractal. The fractal dimensions of the clusters are estimated to be 1.88 in 2 dimensional space and 2.43 in 3 dimensions.

6. CONCLUSION

In the complexity theory is notable involving of the iterative functions in behaviour of fractal pattern. Study of the nonlinear equation, treated into iterative techniques, makes the subject of this paper. It consists in a short revue of the most important principles of the fractal calculus and complexity applications in fundamental sciences and technologies. Were been presented also some new ideas of analysis to iterative relations like as named *the modified logistic equation*. On this relation can be performing some studies with valuable results in numeric analysis area. In bifurcation diagram of logistic map (figure 1) are presented results of the own program run, that is written under C++ programming language.

7. REFERENCES

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