IDENTIFICATION OF STIFFNESS MATRIX FOR LINE TYPE FINITE ELEMENTS WITH VARYING CROSS-SECTIONAL AREA

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Abstract: *The paper provides some aspects regarding the identification of stiffness matrix for line type finite elements with varying cross-sectional area; the element is loaded by an axial load and it is placed in gravitational field. Firstly, the paper presents the practical starting problem: identifying the stresses in a bar from a structure composed by a horizontal beam supported by two vertical bars with varying cross sectional area. It is presented the geometrical modeling (by attaching the local natural reference frame) and then it is written the expression of the potential energy and it is identified the stiffness matrix. Finally there are presented the results obtained from the theoretical modeling and from the finite element modeling by using specialized software.*
Key words: *finite element method, stiffness matrix, bar.*

1. INTRODUCTION

The finite element method is used to analyze stress problems, thermal field problems, buckling problems, eigen frequencies problems etc. The stress analysis problem is based on the variational formulation which is based on the minimization of the potential energy of the system; the aim of the elasticity problem is to identify the stiffness matrix and in a particular way, is to identify the stiffness matrix for each finite element [1, 2, 4]. The study presents a method used to identify the stiffness matrix for a line type finite element with varying cross-sectional area which is modeling the geometrical domain for a bar. The bar it's a part of a mechanical system composed by a horizontal beam *1* supported by two vertical bars (*2* and *3*); the concentrated external load *P* is acting on the beam at the point *C*. Between the in contact bodies there are considered the following constraints: rotational joint between the bars and the horizontal beam and between the bar *3* and the ground; a clamp between the bar *2* and the ground (figure 1).

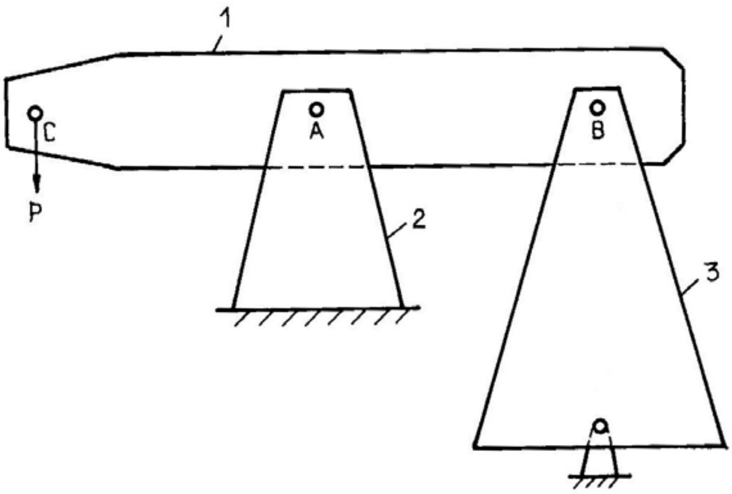

Fig.1 *The mechanical system*

2. THEORETICAL ASPECTS

To obtain the stiffness matrix for the line type finite element with varying cross-sectional area firstly it is done the geometrical modeling (figure 2); A local ξ – natural reference frame [3, 4] is attached to the reference finite element and it is written the approximation function for the linear modeling

$$y = a_0 + a_1\xi \,, \tag{1}$$

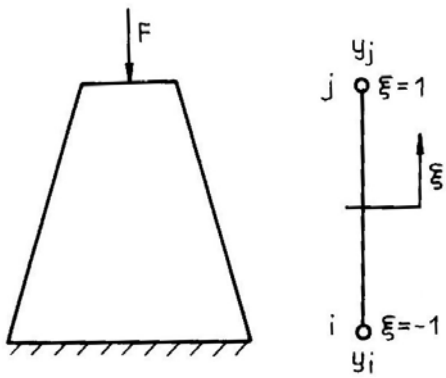

Fig.2 *The bar with varying cross-sectional area*

for y=y_i and y=y_j it results ξ=-1 and, respectively, ξ=1; a_0, a_1 are the unknown polynomial coefficients; i and j are the finite element's nodes. The expression of the approximation function will be

$$y = \frac{1}{2}(1-\xi)y_i + \frac{1}{2}(1+\xi)y_j \tag{2}$$

and it can be written the shape functions [1, 2, 4]

$$N_i = \frac{1}{2}(1-\xi),\ N_j = \frac{1}{2}(1+\xi). \tag{3}$$

The expression for the linear varying cross-sectional area is represented by the equation

$$A = N_i A_i + N_j A_j, \tag{4}$$

where A_i and A_j are the areas for the node i and the node j.

Generally, for the body to be in equilibrium, its potential energy must be a minimum. The general expression for the potential energy is [1, 4]

$$\Pi = \Lambda - W, \tag{5}$$

where Λ represents the strain energy

$$\Lambda = \frac{1}{2}\int_V [\varepsilon]^t [E][\varepsilon] dV \tag{6}$$

and W – the work done by the external loads acting on the body

$$W = [F]^t [d]. \tag{7}$$

The parameters signification from the upper equations is: [ε] – strains matrix; [E] – elasticity matrix; [F] – external loads matrix; [d] – displacements vector.

For a linear modeling of the displacements (with v_i and v_j the nodal displacements)

$$v = N_i v_i + N_j v_j, \tag{8}$$

it can be written the expression

$$[\varepsilon] = \varepsilon_y = \frac{dv}{dy} = \frac{dv}{d\xi}\frac{d\xi}{dy} = \frac{v_j - v_i}{y_j - y_i}. \tag{9}$$

For a homogenous and isotropic material [E]=E.

The differential volume operator is

$$dV = d(Ay) = \frac{1}{2}\left[(A_j y_j - A_i y_i) + \xi(A_i y_i - A_i y_J - A_j y_i + A_j y_j)\right] d\xi. \tag{10}$$

The expression of the potential energy will be

$$\Pi = \frac{1}{4}E\left(\frac{v_j - v_i}{y_j - y_i}\right)^2 \left[\int_{-1}^{1}(A_j y_j - A_i y_i)d\xi + \int_{-1}^{1}\xi(A_i y_i - A_i y_j - A_j y_i + A_j y_j)d\xi\right] - [F]^t[d], \tag{11}$$

or

$$\Pi = \frac{1}{2} E\left(\frac{v_j - v_i}{y_j - y_i}\right)^2 \left(A_j y_j - A_i y_i\right) - [F]^t [d]. \tag{12}$$

For the body to be in equilibrium, its potential energy must be a minimum; so it means Π=min; in mathematical way it means

$$\delta\Pi = \frac{\partial\Pi}{\partial v_i}\delta v_i + \frac{\partial\Pi}{\partial v_j}\delta v_j = 0 \tag{13}$$

and because $\delta v_i \neq 0$ and $\delta v_j \neq 0$ it means

$$\begin{cases} \dfrac{\partial\Pi}{\partial v_i} = 0, \\ \dfrac{\partial\Pi}{\partial v_j} = 0 \end{cases} \tag{14}$$

so

$$\begin{bmatrix} -\dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} & \dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} \\ \dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} & -\dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \end{bmatrix} = \begin{bmatrix} F_i \\ F_j \end{bmatrix} \tag{15}$$

or $[k][d]=[F]$, where $[k]$ is the stiffness matrix; $[d]$ – the displacements matrix; $[F]$ – external loads matrix; so, the expression for the stiffness matrix, the displacements vector and the external loads will be

$$[k] = \begin{bmatrix} -\dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} & \dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} \\ \dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} & -\dfrac{E\left(A_j y_j - A_i y_i\right)}{\left(y_j - y_i\right)^2} \end{bmatrix}; \tag{16}$$

$$[d] = \begin{bmatrix} v_i \\ v_j \end{bmatrix}; \tag{17}$$

$$[F] = \begin{bmatrix} F_i \\ F_j \end{bmatrix}. \tag{18}$$

3. FINITE ELEMENT MODELING

To check the approximation error for the proposed modeling it is done the finite element model (for the three dimensional domain) for the bar with varying cross sectional area presented in figure 2. The finite element model is done by using CATIA V5R10 and it is presented in figure 3. On the upper face of the bar is acting a distributed load.

Fig.3 *The finite element model done in CATIA V5R10*

4. RESULTS

After computation there are obtained the displacements and Von Misses stresses fields which are shown in figure 4.

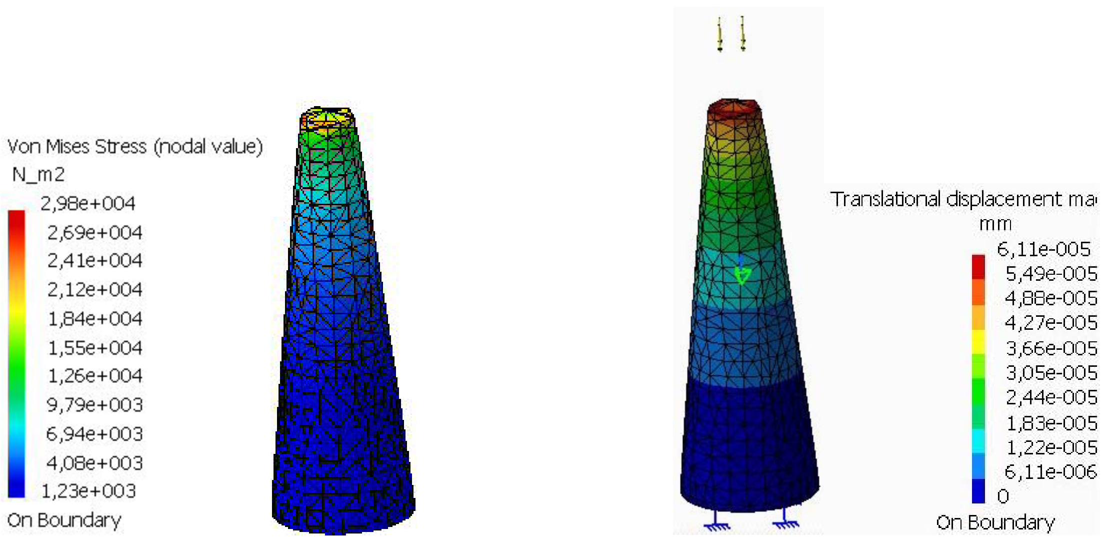

Fig.4 *The results after finite element model computation*

The parameters value used in simulation are: material: steel; applied load: F=500 N; length of the bar: l=2000 mm; cross sectional diameters: d_i=200 mm, d_j=600mm.

The variation of the displacement concerning to the theoretical modeling is presented in figure 5. It can be observed that the maximum value is around 0.00006 mm close to the maximum value shown in figure 4 (0.0000611 mm); so, the error of modeling is small.

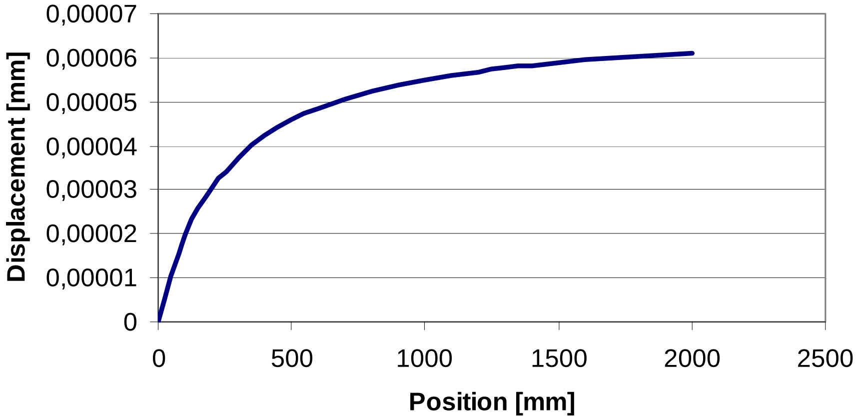

Fig.5 *The displacement variation*

5. CONCLUSIONS AND FURTHER RESEARCH

The main conclusion, from modeling is that the method used in theoretical modeling – the variational method – it could be applied to the one dimensional domains successfully; the results show that, for example, the displacements values obtained from the theoretical modeling and the finite element modeling are close.

For further research it will be considered the case when the variation of cross-sectional area will be a high-order function.

6. REFERENCES

1. Akin, J. E. Finite Element Analysis for Undergraduates. Academic Press, Orlando, Florida, USA, 1996.
2. Fagan, M. J. Finite Element Analysis. Theory and Practice. Longman Singapore Publishers, Australia, 2002.
3. Lates, M. T., Jula, A., Mogan, G. L. Stress analysis for a tripod coupling with cylinder-cylinder contacts using finite elements method. The seventh Yugoslav Tribology Conference YUTRIB'01. Beograd, 11 – 12 october, 2001, p.4.65 – 4.68.
4. Mogan, G. L. Metoda elemenetelor finite in inginerie. Bazele teoretice. Brasov, Editura Lux Libris, 1997.