A METHOD FOR OPTIMUM DESIGN OF BEAMS WITH DIFFERENT CROSS SECTION SHAPE.
Part II constraint functions, examples

Filip Văcărescu Daniela, associate prof.Ph.D., North University of Baia Mare, Romania

Abstract: *A beam clamped at both ends has been optimized to show the validity of proposed concept. The optimization procedure results in the final cross-sectional dimensions instead of cross-sectional properties like area, moment of inertia an rotational moment of inertia, and no further treatment has to be done.*

The designer may compare the results for different cross-sectional shapes and select the best. For instance, the Z cross sectional shape is the best for clamped on both ends studied [3].
Key words: *optimization procedure results, results for different cross-sectional shapes*

4. THE TREATMENT OF OBJECTIVE AND CONSTRAINT FUNCTIONS

4.1 Objective function and its derivate

The objective function W(t) can be written [4]

$$W(t) = \sum_{e=1}^{NB} \rho_e L_e A_e \tag{30}$$

where the summation is done over all beams, ρ_e the material density of beam L_e the length, A_e the cross-sectional area (a function of the design variables t_j).

Differentiate eqn (30) with respect to t_j

$$\frac{\partial W}{\partial t_j} = \sum_{e=1}^{NB} \rho_e L_e \frac{\partial A_e}{\partial t_j} \tag{31}$$

where $\frac{\partial A_e}{\partial t_j} \neq 0$ only if t_j is a design variable within the e element

$$\frac{\partial A_e}{\partial t_j} = \frac{A_e\left(t_j + \Delta t_j\right) - A_e\left(t_j\right)}{\Delta t_j} \tag{32}$$

A_e ($t_j + \Delta t_j$) and A_e (t_j) are derived by the finite element program.

4.2 Displacement and derivatives

Nodal displacements and rotations are determined by solving the equilibrium equations

$$KU = P \tag{33}$$

where: K is the structural stiffness matrix

U is the displacement matrix

P is the load matrix

A specific displacement d can be express as a linear combination of U:

$$d = q^T \cdot U \tag{34}$$

where q is a fixed user specified vector used to select degree of freedom which are to be constrained.

Differentiate eqn (34) with respect to t_j and consider that q is constant:

$$\frac{\partial d}{\partial t_j} = q^T \cdot \frac{\partial U}{\partial t_j} \tag{35}$$

Now, differentiate (33) and solve it for $\frac{\partial U}{\partial t_j}$:

$$\frac{\partial U}{\partial t_j} = -K^{-1}\frac{\partial K}{\partial t_j}U \tag{36}$$

where K is assembled by the contributions from the different element and it can be thus write

$$\frac{\partial U}{\partial t_j} = -K^{-1}\left(\sum_e \frac{\partial K_e}{\partial t_j}U_e\right) \tag{37}$$

$\frac{\partial K_e}{\partial t_j} \neq 0$ only if tj is a design variable linked to element e. With

$$q_p = -\sum_e \frac{\partial K_e}{\partial t_j}U_e \tag{38}$$

can write

$$K\frac{\partial K_e}{\partial t_j} = q_p \tag{39}$$

Simply, get the $\frac{\partial U}{\partial t_j}$ by solving the eqn (39) q_p is called the pseudo load [3] and for each variable and each loading case we can obtained the right hand side in eqn (39).

Eqn (35) can be written using (36) and (37) as:

$$\frac{\partial d}{\partial t_j} = q^T K^{-1} \frac{\partial K}{\partial t_j} U = -U^T \frac{\partial K}{\partial t_j} K^{-1} q = -\sum_e U_e^T \frac{\partial K_e}{\partial t_j} V_e \tag{40}$$

where V_e is the part of the virtual displacement V associate with element e. V result by solving eqn (41):

$$KV = q \tag{41}$$

where q is the vector in eqn (34) and is called the virtual load [3].

The element stiffness matrices are derived numerically:

$$\frac{\partial K_e}{\partial t_j} = \frac{K_e\left(t_j + \Delta t_j\right) - K_e\left(t_j\right)}{\Delta t_j} \tag{42}$$

where K_e ($t_j + \Delta t_j$) and K_e (t_j) are derived by the finite element method.

The solution of eqn (33) results in a decomposition of K. The solution of eqn (39) or (41) then only requires a backsubtitution to get $\frac{\partial U}{\partial t_j}$ or V.

4.3 Stress and derivatives

Two approximation concepts for the stress derivatives is considered:

a) full approximation;

b) simplified approximation with invariant nodal forces.

a) Full approximation. The beam stress in a certain point one of the three cross-sections is given by eqn (12) and (1)

$$\sigma = fF = fq_e^T U_e = q_e^{''T} U_e \tag{43}$$

$$q_e^{''T} = fq_e^T \tag{44}$$

Differentiate eqn (43) with respect to t_j:

$$\frac{\partial \sigma}{\partial t_j} = \frac{\partial q_e^{''T}}{\partial t_j} U_e + q_e^{''} \frac{\partial U_e}{\partial t_j} \tag{45}$$

where

$$\frac{\partial q_e^{''T}}{\partial t_j} = \frac{q_e^{''T}\left(t_j + \Delta t_j\right) - q_e^{''T}\left(t_j\right)}{\Delta t_j} \tag{46}$$

and $\frac{\partial U_e}{\partial t_j}$ is given by eqn (37) if we are using the pseudo load technique [3].

Differentiation of Tsai factor eqn (13) gives:

$$\frac{\partial T_s}{\partial t_j} = \left(\frac{\partial \sigma_x}{\partial t_j} \frac{\partial \tau_{xy}}{\partial t_j} \frac{\partial \tau_{xz}}{\partial t_j} \right) \begin{Bmatrix} \frac{\sigma_x}{\sigma_a^2} \\ \frac{\tau_{xy}}{\tau_a^2} \\ \frac{\tau_{xz}}{\tau_a^2} \end{Bmatrix} \frac{1}{T_s} = \left(\frac{\partial \sigma}{\partial t_j} \right)^T \begin{Bmatrix} \frac{\sigma_x}{\sigma_a^2} \\ \frac{\tau_{xy}}{\tau_a^2} \\ \frac{\tau_{xz}}{\tau_a^2} \end{Bmatrix} \frac{1}{T_s} \tag{47}$$

The virtual load technique will be use instead .

Transpose eqn (43):

$$\sigma^T = U_e{}^T q_e'' \tag{48}$$

and differentiate

$$\frac{\partial \sigma^T}{\partial t_j} = U_e^T \frac{\partial q''}{\partial t_j} + \frac{\partial U_e^T}{\partial t_j} q'' \tag{49}$$

Use eqn (36):

$$\frac{\partial \sigma^T}{\partial t_j} = U_e^T \frac{\partial q''}{\partial t_j} - U^T \frac{\partial K}{\partial t_j} K^{-1} q'' = U_e^T \frac{\partial q''}{\partial t_j} - \sum_e U_e^T \frac{\partial K_e}{\partial t_j} V_e \tag{50}$$

where V_e is the part of the virtual displacement V associated with element e. V is obtained by solving eqn (51):

$$KV = q'' \tag{51}$$

where q'' is q_e' expressed with zeros to the global vector size. q'' is the virtual load associated with the stress constraints. Equation (47) is used to get the derivatives of the Tsai factor. Each q'' vector is a 12×3 matrix and will get tree right sides in eqn (51) for each stress constraint. It will be $n_s = 3 \times NSC$ right hand sides in the virtual load technique.

b) Simplified approximation with invariant nodal forces.

Simply assume that the nodal forces will not vary as the design variables change within a sub problem, but that the nodal forces will be modified after the solution of each sub problems, which means $\frac{\partial F}{\partial t_j} = 0$

Differentiation of eqn (12) will then give:

$$\frac{\partial \sigma}{\partial t_j} = \frac{\partial f}{\partial t_j} F \tag{52}$$

where

$$\frac{\partial f}{\partial t_j} = \frac{f\left(t_j + \Delta t_j\right) - f\left(t_j\right)}{\Delta t_j} \tag{53}$$

and again, using eqn (47) will obtain the derivatives of Tsai factor.

The approximation does not involve neither pseudo loads nor virtual load. Therefore must solve neither eqn (37) nor eqn (51) to get the derivatives. The number of right hand sides in the pseudo load and virtual load equations for a few cases are shown in table 1.

Table 1. Number of right hand sides with full approximation for stress, and simplified approximation for stress

Constraints	Full approximation		Simplified approximation	
	Pseudo load	Virtual load	Pseudo load	Virtual load
Only displacements	NIDV × NLC	NDC	NIDV × NLC	NDC
Only stress	NIDV × NLC	3 × NSC	0	0
Displacements and stresses	NIDV × NLC	NDC + 3 × NSC	NIDV × NLC	NDC

5. BEAM CLAMPED AT BOTH END

The validity of the proposed concept is traded in [3] for a beam clamped at both ends (fig. 2)

The loading condition and the dimension are shown. Design data: $\rho = 2{,}8 \cdot 10^3$ kg/m; $E = 7{,}06 \cdot 10^{10}$ N/m^2; $G = 2{,}7 \cdot 10^{10}$ N/m^2; $\sigma_a = 4{,}6 \cdot 10^8$ N/m^2; $\tau_a = 2{,}65 \cdot 10^8$ N/m^2; $P_y = 6{,}86 \cdot 10^5$ N; $P_z = 4{,}9 \cdot 10^5$ N; dy = 0,04 m; dz = 0,04 m (at the midpoint); $t_1 = t_2 = 0{,}05$ m for each element ($t_2 = 0{,}01$ m for tube cross-section) $t_3 = t_4 = 0{,}01$; $\overline{t_j} = 2{,}0$ (j = 1,4).

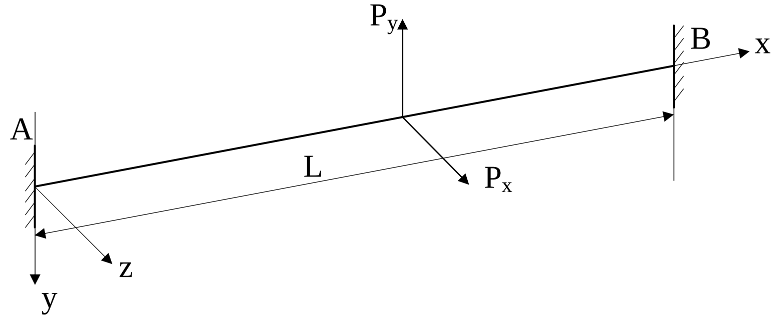


Fig. 2. Beam clamped at both ends

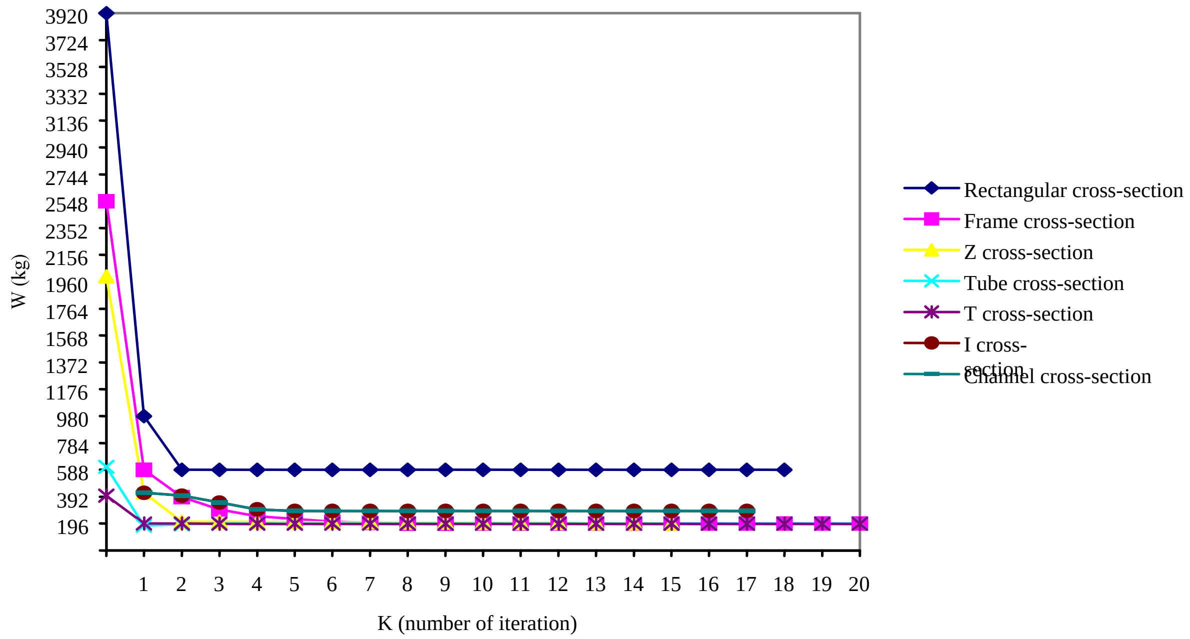
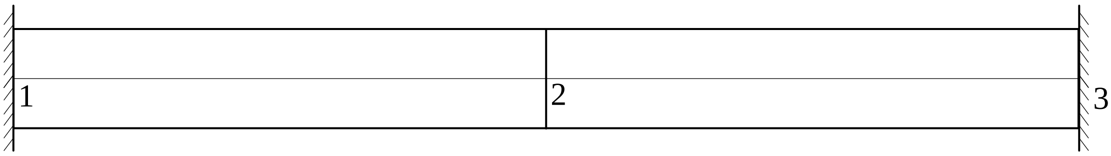


Fig 4. Weight history of beam clamped at both ends for different cross sectional shape

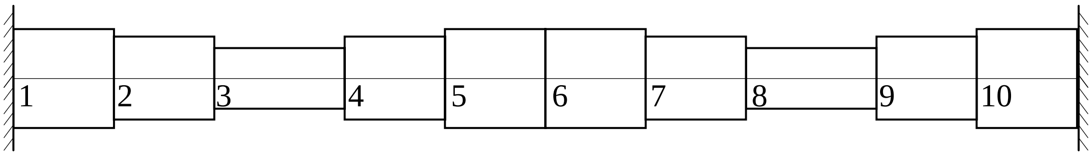
The beam was divided into 2, 10 and 20 elements. The weight history for the clamped beam with 10 elements for different cross-section shape is shown in fig. 4 and Table 3. The optimal design shape (for the tube cross-section) for three kinds of dispersed models are in fig 5.

Table 3. Iteration history data for clamped beam with different cross-sectional shape

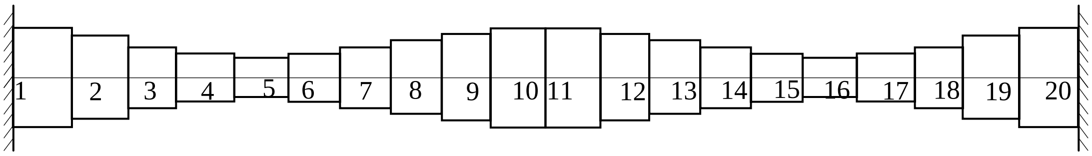
Number of analysis	Weight (kg)						
	Type of cross-section shape						
	Rectangular	Frame	Z	Tube	T	I	Channel
0	3920.0	2508.8	2038.4	694.57	456.29	2038.4	2038.4
1	980.0	627.2	509.6	173.64	220.65	509.6	509.6
2	594.49	366.11	279.14	182.54	188.89	397.46	358.01
3	611.74	305.47	232.86	171.95	188.05	353.25	312.63
4	625.65	271.76	204.47	165.45	163.87	319.20	282.91
5	617.56	246.06	181.16	165.17	172.65	293.25	257.75
6	623.05	226.25	163.36	165.10	160.13	272.83	238.87
7	623.07	210.54	149.22	165.01	171.02	255.26	222.73
8	621.64	197.70	137.93	165.04	163.82	240.49	209.32
9	619.31	186.96	128.56	165.03	170.31	228.07	197.96
10	625.56	186.26	127.28	165.02	166.20	221.28	192.73
11	622.72	186.26	127.52	165.02	169.56	215.80	187.93
12	622.10	186.26	127.57	165.05	168.57	211.15	185.47
13	625.76		127.58	165.08	170.30	206.86	183.65
14	624.13		127.61	165.10	169.34	204.04	183.27
15	623.74		127.62	165.13	170.75	202.02	183.35
16	625.91			165.14	169.90	201.45	183.07
17	625.14			165.16	171.12	201.38	
18	626.04			165.17	170.34		
19				165.19	171.42		
20				165.19	170.70		



1. 2 element W = 200.10 kg



2. 10 element W = 165.19 kg



3. 20 element W = 140.98 kg

Fig. 5. Optimum shapes of beam (tube cross-section)

Some comparation of results are illustrated in table 4 and 5 [3]

Table 4. Comparations

Z cross section		Full approximation		Simplified approximation	
Number of analysis		7	7	7	7
Final weight (kg)		126,6	126,6	126,6	126,6
Active constrains	Displacement	2	2	2	2
	Stresses	1	1	1	1

Table 5. The lower limits of the design variables

Z cross section	Number of analysis	
	10	7
Side lower limits (cm)	0,5	1,0
Final weight (kg)	80,8	126,6

6. DISCUSSION AND CONCLUSIONS

The optimization procedure results in the final cross section instead of cross-section properties like area, moment of inertia, and rotational moment of inertia and no further treatment has to be done.

The designer may compare the results for different cross-sectional shape and select the best. For the beam in fig 3 the z cross-sectional shape, for instant, is the best example (table 4, 5) [3].

The full stress approximation concept gives the same convergence speed. The simplified stress approximation gives a lover cost per iteration with the virtual load technique.

For further development it will be interesting to take account the effect of local buckling.

REFERENCES

[1] Cingini, C., Contro, R., Optimal design of beams discretiyed by elastic plastic finite element. Computers & Structures 20, 475-478 (1985)

[2] Ding, Y., Esping, B., J., D., Optimum design of beams, Computers & Structures 24 1986 707-721

[3] Pederseu, P., Jongensen, L., Minimum mass design of elastic frames subjected to multiple load casses, Computers & Structures 18 1984 147-157

[4] Prasad, B., Explicit constraint approximation forms in structural optimization - part. 1: analyses and projections, Computers Method application Mechanical Engenieering 40 1983 1-26