A METHOD FOR OPTIMUM DESIGN OF BEAMS WITH DIFFERENT CROSS SECTION SHAPE.
Part I, design problem, optimization problem and solution method

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Abstract: *The minimum weight design problem for frame structures subject to stress and displacement constrains is treated. The cross sectional dimensions are used as design variables and the hybrid approximation technique in combination with the dual method from mathematical programming is used. Seven different sectional shapes are treated.*
Key words: *minimum weight design problem, hybrid approximation technique.*

1. INTRODUCTION

The goal is to solve the minimum weight design problem for frame structures when stress and displacement constraints are considered Seven cross sectional shape: rectangular, frame, tube Z, I, T and channel section are treated.

In this paper the cross sectional dimensions is use as primal design variables. Often the cross sectional properties such as area, moments of inertia and rotational moments of inertia are use as primal design variables. First order approximations of the constraints with respect to those variables or to their reciprocals are supposed to give a better approximations with respect to the cross-sectional dimensions or their reciprocals. Once the cross sectional properties are determined the cross-sectional dimensions can be determinate by solving a second level optimization problem for each beam.

The use of the cross-sectional dimensions as design variables is combine with the hybrid approximation of the objective and constraint functions proposed previously [4]. The hybrid method will always use the most conservative approximation of a first-order approximation in the design variables or their reciprocals.

So it's created a sequence of conservative, convex first-order problems. Each problem is solved using the dual technique from mathematical programming. Two different first-order approximations for the stress was considered, and it will also use both the pseudo and virtual load techniques for the constraint derivatives.

2. BEAM ELEMENT AND DESIGN VARIABLES

2.1 Cross-sectional forces and stiffness matrix.

The cross-sectional force vector $F = (F_x, F_y, F_z, M_x, M_y, M_z,)^T$ in the space beam element *e* shown in figure 1 can be written (F is a function of x coordinate):

$$F = T \cdot F' = TA_bK_{e'}U_e = q_e^T U_e \qquad (1)$$

where:

$$q_e^T = TA_bK_{e'} \qquad (2)$$

F and F' are the cross-sectional force vectors (6 × 1) expressed in the local and global coordinate systems, respectively

T is a coordinate transformation matrix (6 × 6)

K_e the stiffness matrix (12 × 12) of the beam element expressed in the global coordinate system

A_b the transformation matrix (6 × 12) from the nodal forces to the cross-sectional forces

U_e the nodal displacement vector (12 × 1) associated with element *e* expressed in the global coordinate system

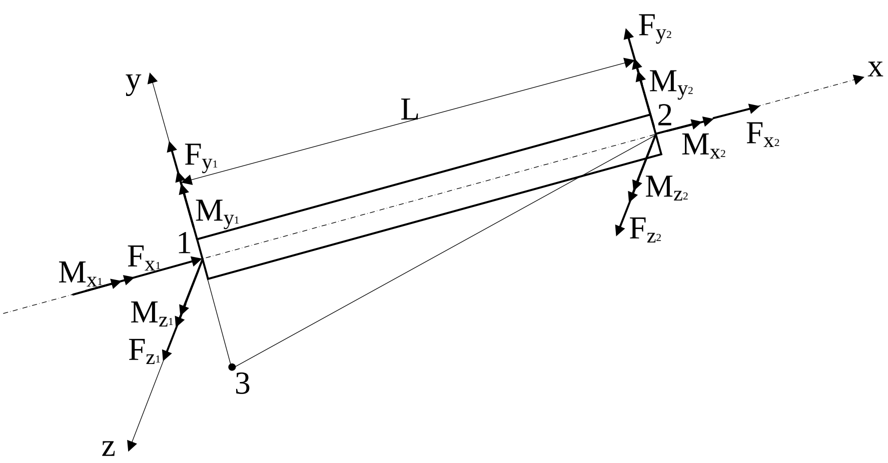

Fig. 1. Space beam element

The preceding matrices can be expressed as:

a) Coordinate transformation matrix:

$$T = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \tag{3}$$

where:

$$S = \begin{pmatrix} S1^T \\ S2^T \\ S3^T \end{pmatrix} \tag{4}$$

$$S_1 = \frac{1}{L12} \begin{Bmatrix} x & 2 & 1 \\ y & 2 & 1 \\ z & 2 & 1 \end{Bmatrix} \tag{5}$$

$$S_3 = \frac{1}{\left\| S1 \times S31 \right\|} S1 \times S31 \tag{6}$$

$$S31 = \begin{Bmatrix} x & 3 & 1 \\ y & 3 & 1 \\ z & 3 & 1 \end{Bmatrix} \tag{7}$$

$$S2 = S1 \times S3 \tag{8}$$

where S1 is the unit vector going from node 1 to node 2

S31 is the vector going from node 1 to the reference node 3

b) Transformation matrix A_b

$$A_b = \begin{bmatrix} a & & & & & & b & & & & & \\ & a & & 0 & & & & b & & 0 & & \\ & & a & & & & & & b & & & \\ & & & a & & & & & & b & & \\ & & 0 & & a & & & & 0 & & b & \\ & & & & & a & & & & & & b \end{bmatrix} \tag{9}$$

where: $a = 1 - \frac{x}{L}$ $b = -\frac{x}{L}$

In the following section we will evaluate F at x = 0, $\frac{L}{2}$ and L, where L is the length of the beam element.

c) Stiffness matrix of the space beam element:

$$K_{e'} = HK_e H^T \tag{10}$$

where: $H = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$

$$K_e = \begin{bmatrix} \frac{EA_e}{L} & & & & & & & & & & & \\ 0 & \frac{12EI_z}{L} & & & & & & & & & & \\ 0 & 0 & \frac{12EI_y}{L} & & & & & & & & & \\ 0 & 0 & 0 & \frac{GJ_p}{L} & & & & & & & & \\ 0 & 0 & -\frac{6EI_y}{L} & 0 & \frac{4EI_y}{L} & & & & & & & \\ 0 & \frac{6EI_z}{L} & 0 & 0 & 0 & \frac{4EI_z}{L} & & & & & & \\ -\frac{EA_e}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_e}{L} & & & & & \\ 0 & -\frac{12EI_z}{L} & 0 & 0 & 0 & -\frac{6EI_z}{L} & 0 & \frac{12EI_z}{L} & & & & \\ 0 & 0 & -\frac{12EI_y}{L} & 0 & \frac{6EI_y}{L} & 0 & 0 & 0 & \frac{12EI_y}{L} & & & \\ 0 & 0 & 0 & -\frac{GJ_p}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_p}{L} & & \\ 0 & 0 & -\frac{6EI_y}{L} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L} & 0 & \frac{4EI_y}{L} & \\ 0 & \frac{6EI_z}{L} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \quad (11)$$

In relation (11):

E and G are the elastic moduli in tension and shear; A_e is the cross-sectional area; I_y and I_z are the moment of inertia; J_p is the rotational moment of inertia.

The expression of A_e, I_y I_z and J_p was calculate for each cross section present in figure 2.

2.2 Cross-sectional shapes and stress points.

Seven kinds of beam cross-sectional shape will be considered (fig. 2) Constrains on the stress at certain points wil be consider. In each cross-section the stresses are calculated in 5-8 points. The number of points (NPNT) is depending on the cross-sectional shape.

The stresses of each point can be written:

$$\sigma = \begin{Bmatrix} \sigma_x \\ \tau_{xy} \\ \tau_{xz} \end{Bmatrix} = fF \quad (12)$$

where $\{\sigma\}$ is the stress vector

f is a (3×6) matrix depending on the cross-sectional domension.

There will be NPNT *f* matrices in each cross-section. The *f* matrices for sever kinds of cross sectional shapes was calculated. The factor used for the stress constrains is the Tsai (T_s) factor, a normalized equivalent stress:

$$T_s = \sqrt{\left(\frac{\sigma_x}{\sigma_a}\right)^2 + \frac{1}{\tau_a^2}\left(\tau_{xy}^2 + \tau_{xz}^2\right)} \tag{13}$$

where σ_a and τ_a are the allowable stress in tension and shear, respectively

2.3 Design variables

The cross sectional dimensions t_j in each beam element will be the design variables (fig. 2)

Usually a number of elements will be forced to have the same dimensions. The design variables t_j is one of those linked beams will then be selected as independent design variables and the others will be linked to these.

The number of beams (NB) will be linked to number of independent beams (NIB) and altogether will be number of independent design variables (NIDV). The cross-sectional properties A_e, I_z, I_y, J_p etc., can be expressed in the cross-sectional dimensions t_j.

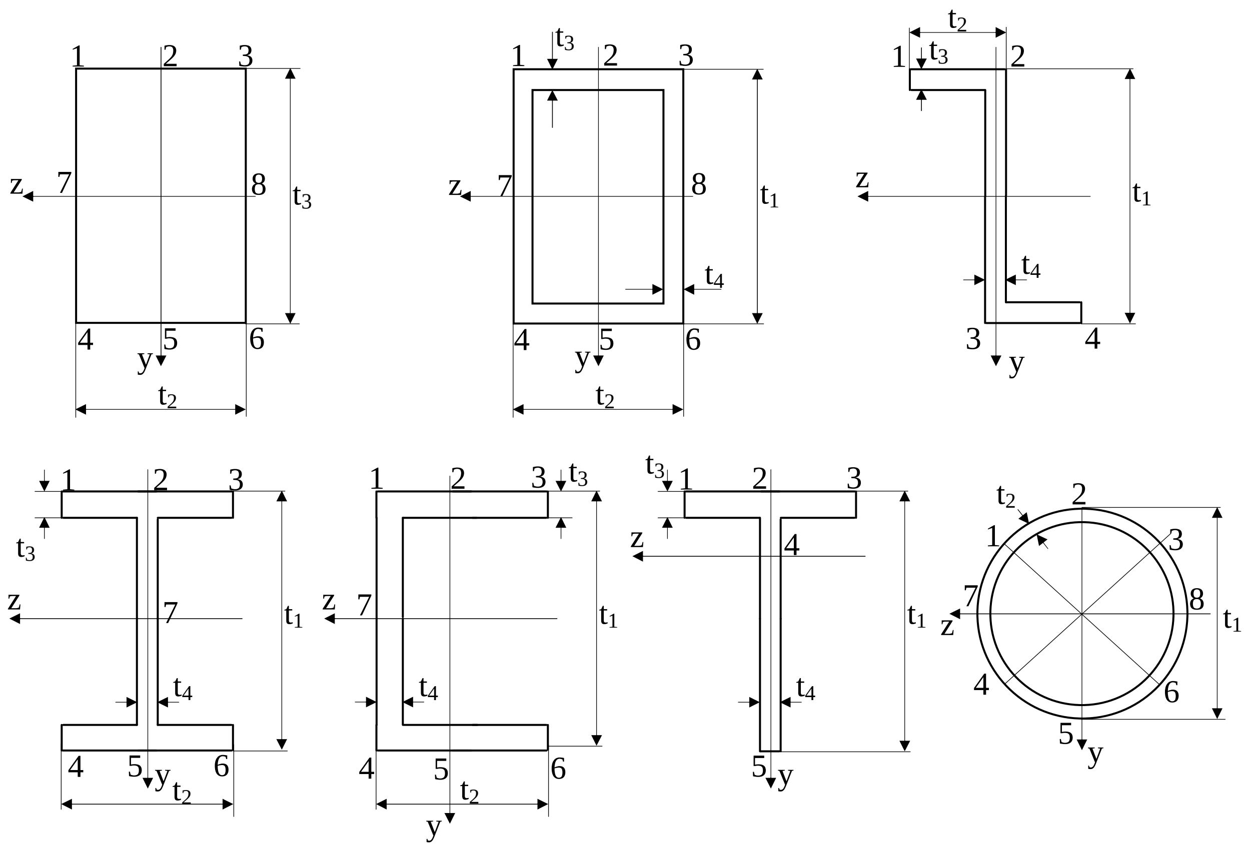

Fig. 2. Seven cross-sectional shapes and stress points

3. OPTIMIZATION PROBLEM AND SOLUTION METHOD

3.1 Problem formulation

The optimization problem P can be state as:

Find the cross-sectional dimensions, where the cross sectional shape is given, then minimized the structural weight and satisfied stress, displacement and side constraints.

It can be expressed mathematically as:

P: min W(t)

$$\text{s.t.} \quad T_{s_j}(t) \le 1 \qquad i = 1, NSC$$

$$\frac{d_k(t)}{\bar{d}_k} \le 1 \qquad k = 1, NDC \tag{14}$$

$$t_j^L \le t_j \le t_j^U \qquad j = 1, NIDV$$

where: W is the structural weight; T_s stress constraints in the ith; d_k displacement constraints in the kth; $\bar{d}_k$ upper limit; t_j^L, t_j^U are the lower and upper values of the independent variable t_j ; NSC is the number of stress constraints; NDC is the number of displacement constraints; NIDV is the number of independent design variables.

Originally exist one stress constraint for each stress point (NPNT - number of points) for each section [3] for each loading case (NLC - number of loading cases), for each beam (NB - number of beams) i.e. NPNT × 3 × NLC × NB.

The number will be reduced by selecting only one constraint for each independent beam so these will be altogether only NIB stress constraints, i.e. NSC = NIB. This strong reduction may lead to instabilities for the optimizer and will than result in slow convergence.

3.2 Approximate problem generation

The problem P can be written in a standard form as:

P: min W(t)

$$\text{s.t.} \quad g_i(t) \le 1 \qquad i = 1, NC$$

$$t_j^L \le t_j \le t_j^U \qquad j = 1, NINV$$

where NC is the number of constraint functions.

The general structural optimal problem P is implicit and nonlinear. In order to solve it, will use the $\bar{P}(k)$ sub problem and each sub problem $\bar{P}(k)$ is first order approximation of the problem P around the design point $t^{(k)}$. The approximation concept is taken from [2]

$\bar{P}(k)$: min $\overline{W}(t)$

$$\text{s.t.} \quad \bar{g}_i(t) \le \underline{g}_i' \qquad i = 1, NC$$

$$\underline{t}_i \le t_j \le \bar{t}_j \qquad j = 1, NIDV \tag{16}$$

where:

$$\overline{W}(t) = \sum_{+}\left(\frac{\partial W}{\partial t_j}\right)^{(k)} t_j - \sum_{-}\left(\frac{\partial W}{\partial t_j}\right)^{(k)} \frac{\left(t_j^{(k)}\right)^2}{t_j} \tag{17}$$

$$\overline{g}_i(t) = \sum_{+}\left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} t_j - \sum_{-}\left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} \frac{\left(t_j^{(k)}\right)^2}{t_j} \tag{18}$$

$$g_i^{'} = 1 - g_i\left(t_j^{(k)}\right) - \sum_{+}\left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} t_j^{(k)} - \sum_{-}\left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} t_j^{(k)} \tag{19}$$

where $\sum_{+}\left(\sum_{-}\right)$ means summation over all positive (negative) determinative terms.

The move limits $\underline{t}_j$ and $\overline{t}_j$ are

$$\underline{t}_j = \max\left(\frac{t_j^{(k)}}{2}, t_j^L\right)$$

$$\overline{t}_j = \left(2 \cdot t_j^{(k)}, t_j^U\right) \tag{20}$$

3.3 Dual solution method

The sub problem $\overline{P}(k)$ is solved by using the duality theory for convex programming. The Lagrangian L is:

$$L\,(t,\lambda) = \overline{W}(t) + \sum \lambda_i \left(\overline{g}_i(t) - g_i^{'}\right) \tag{21}$$

The dual objective function, ϕ:

$$\phi\,(\lambda) = \min_t L(t,\lambda) \tag{22}$$

where ϕ is concave in λ. Problem P(k) has its optimum where the dual function $\phi(\lambda)$ has its maximum,

$\max_i \; \phi(\lambda)$

s.t. $\lambda_i > 0$ i = 1, NC

L is separable and a differentiation of (21) with respect to t_j, due to (22), gives $t_j = t_j(\lambda)$

$$\frac{\partial L}{\partial t_j} = 0$$

with

$$f_j = \left(\frac{\partial W}{\partial t_j}\right)^{(k)}$$

$$P_j = \sum_i \lambda_i \left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} \text{ for } \frac{\partial g_i}{\partial t_j} > 0 \tag{24}$$

$$Q_j = -\sum_i \lambda_i \left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} t_j^{(k)} \text{ for } \frac{\partial g_i}{\partial t_j} < 0$$

inplies for $f_j > 0$

$$f_j + P_j - \frac{1}{t_j^2} Q_j = 0 \tag{25}$$

following solutions

$$t_j = \left(\frac{Q_j}{f_j + P_j}\right)^{\frac{1}{2}} \text{ for } \underline{t}_j^2 < \frac{Q_j}{f_j + P_j} < \bar{t}_j^2$$

$$t_j = \underline{t}_j \text{ for } Q_j \le \underline{t}_j^2 (f_j + P_j) \tag{26}$$

$$t_j = \bar{t}_j \text{ for } Q_j \ge \bar{t}_j^2 (f_j + P_j) \tag{26}$$

for $f_j < 0$

$$\frac{1}{t_j^2} f_j (t_j^{(k)})^2 - \frac{1}{t_j^2} Q_j + P_j = 0 \tag{27}$$

we get:

$$t_j = \left(\frac{Q_j - f_j (t_j^{(k)})^2}{P_j}\right) \text{ for } \underline{t}_j^2 < \frac{Q_j - f_j (t_j^{(k)})^2}{P_j} < \bar{t}_j^2$$

$$t_j = \underline{t}_j \text{ for } P_j \underline{t}_j^2 + f_j \left(t_j^{(k)}\right)^2 \ge Q_j \tag{28}$$

$$t_j = \bar{t}_j \text{ for } P_j \bar{t}_j^2 + f_j \left(t_j^{(k)}\right)^2 \le Q_j$$

In order to solve the dual problem (23), also need the derivatives of $\phi(\lambda)$, which are given by:

$$\frac{\partial \phi}{\partial \lambda_i} = \sum_+ \left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} t_j - \sum_- \left(\frac{\partial g_i}{\partial t_j}\right)^{(k)} \frac{\left(t_j^{(k)}\right)^2}{t_j} - \bar{g}_i \tag{29}$$

The dual problem (23)is solved by an appropriate routine from some standard mathematical program library.

REFERENCES

[1] Cingini, C., Contro, R., Optimal design of beams discretiyed by elastic plastic finite element. Computers & Structures 20, 475-478 (1985)

[2] Ding, Y., Esping, B., J., D., Optimum design of beams, Computers & Structures 24 1986 707-721

[3] Pederseu, P., Jongensen, L., Minimum mass design of elastic frames subjected to multiple load casses, Computers & Structures 18 1984 147-157